where

$$N = \ln\left\{\frac{-1 + (1+Q^2)\eta_f + (1+Q^2)^{1/2}\,[(1+Q^2)\eta_f^2 - 2\eta_f + 1]^{1/2}}{-1 + (1+Q^2)\eta_i + (1+Q^2)^{1/2}\,[(1+Q^2)\eta_i^2 - 2\eta_i + 1]^{1/2}}\right\} \tag{25a}$$

Equation (25a) has been computed and tabulated for $Q = 1/(3)^{1/2}$, 1, $(3)^{1/2}$ by Nyland.[6] See also Ref. 1.

Equation (25) is overlaid in Fig. 2 as lines of constant L/D. A particular value of N occurs at the maximum latitude, $\varphi = \varphi_*$, such that

$$N = -2\pi/(L/D)$$

For this case, Fig. 3 displays the L/D required as a function of initial speed. The validity of the assumptions in Eq. (1) begins to break down in the final phase of the trajectory, and, although zero final velocity has been used in this analysis, the effect on range is small (<10%). As seen from Fig. 3, the L/D required to reach the maximum latitude φ_* decreases monotonically as the maneuver initiation speed is increased. It also is noted that the required L/D for a given value of the maneuver index Q is the same as for the reciprocal of Q.

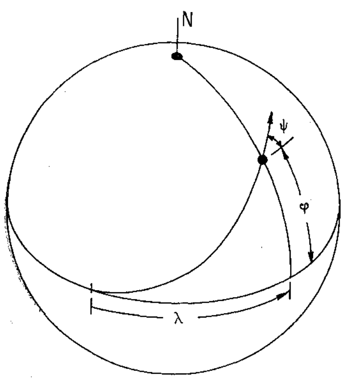

Fig. 1a Equatorial tangent minor circle coordinates.

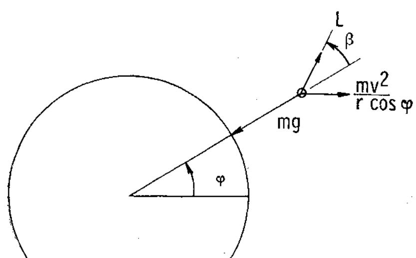

Fig. 1b Polar minor circle force relationships.

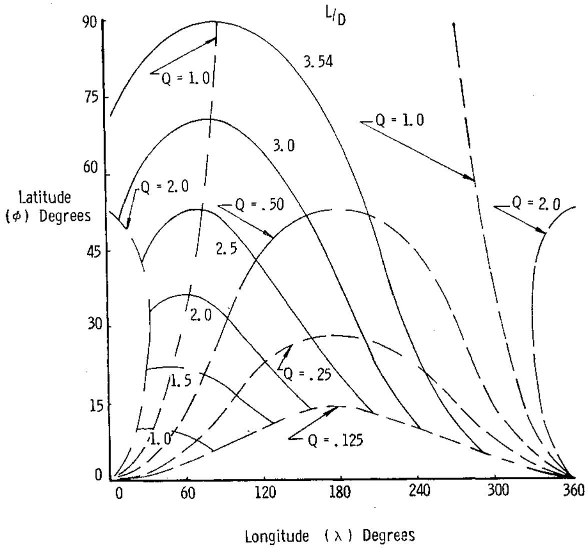

Fig. 2 Equatorial tangent minor circle maneuver for $\eta_i = 1.0, \eta_f = 0$.

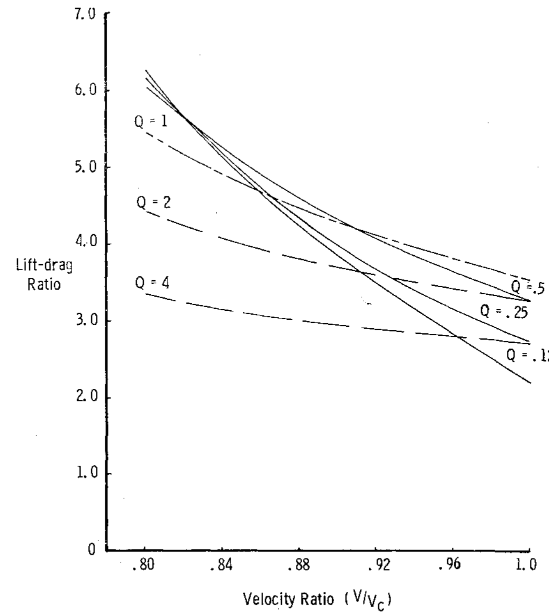

Fig. 3 Required L/D to attain the maximum lateral offset, φ_*, as a function of initial speed.

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Stresses in Solid Propellants Due to High Axial Acceleration

JAMES R. HUTCHINSON*

Lockheed Missiles and Space Company, Sunnyvale, Calif.

Nomenclature

G	=	shear modulus
u,v,w	=	displacements in x, y, and z directions, respectively
x,y,z	=	Cartesian coordinate system
z	=	axial coordinate, positive in forward direction of motor
Z	=	body force in z direction
β	=	acceleration loading
γ	=	shear strain
ϵ	=	direct strain
ρ	=	weight density
σ	=	direct stress
τ	=	shear stress
ϕ	=	stress function

Received July 26, 1963. The author wishes to thank E. H. Lee Lockheed Consultant, for his helpful suggestions.

* Research Specialist, Mechanical and Mathematical Sciences Laboratory.

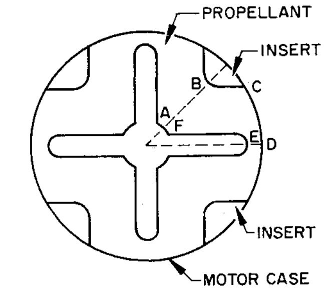

Fig. 1 High-acceleration rocket: cross section.

Introduction

HIGH-ACCELERATION solid propellant rockets have recently become the subject of particular interest in rocket technology. In these vehicles, the shear stresses between the propellant and bond, caused by high axial accelerations, have become an important problem.

A typical cross section of one of these high-acceleration rockets is shown in Fig. 1. The inserts in the propellant serve to reduce both the time taken for thrust tail-off and the total vehicle weight. In previous analyses,[1] the rocket has been treated as a thick-walled, circular cylinder with a rigid motor case. For the more usual configuration, such as shown in Fig. 2, this treatment is a good approximation; however, it is a poor approximation for the configuration in Fig. 1.

According to Saint-Venant's principle, if the ratio of the length of the rocket to its external diameter is sufficiently large (approximately 3), the end effects will be negligible midway between the ends of the motor, and the shear stresses at that location will be the same as in an infinitely long rocket motor. Also, if the end closures are bonded to the propellant, the shear stresses will diminish near the ends. Therefore, solving for the stresses on the basis of an infinitely long cylinder is justified.

Generally, the inserts are stiffer than the propellant. If they are considered rigid, the shear stresses in the propellant will be higher than they would be if the inserts were considered deformable; hence, to assume that the inserts are rigid will lead to a conservative design.

Solution

The rigorous solution for the shear stresses presented here is based upon the following assumptions: 1) the cylinder (Fig. 1) is infinitely long, 2) the propellant is an elastic or linear viscoelastic material, 3) the inserts and the motor case are rigid, and 4) the cylinder is being accelerated axially.

Using the semi-inverse method, it is assumed that

$$u = v = 0 \qquad w = w(x,y) \tag{1}$$

Then, from the strain displacement relationships, the strain components become

$$\epsilon_x = \epsilon_y = \epsilon_z = \gamma_{xy} = 0$$

$$\gamma_{xz} = \partial w/\partial x \qquad \gamma_{yz} = \partial w/\partial y$$

From Hooke's law (assuming elastic behavior), the stresses are

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$$

and

$$\begin{aligned} \tau_{xz} &= G\gamma_{xz} = G(\partial w/\partial x) \\ \tau_{yz} &= G\gamma_{yz} = G(\partial w/\partial y) \end{aligned} \tag{2}$$

There is only one equilibrium condition, namely,

$$(\partial\tau_{xz}/\partial x) + (\partial\tau_{yz}/\partial y) + Z = 0 \tag{3}$$

The body force is given by the expression

$$Z = -\rho\beta \tag{4}$$

Substitution of Eqs. (2) and (4) into Eq. (3) gives

$$G[(\partial^2 w/\partial x^2) + (\partial^2 w/\partial y^2)] = \rho\beta \tag{5}$$

Introduction of the definition

$$\phi = wG/\rho\beta \tag{6}$$

and substitution of this into Eqs. (5) and (2), respectively, gives

$$(\partial^2\phi/\partial x^2) + (\partial^2\phi/\partial y^2) = 1 \tag{7}$$

$$\tau_{xz} = \rho\beta(\partial\phi/\partial x) \tag{8}$$

$$\tau_{yz} = \rho\beta(\partial\phi/\partial y) \tag{9}$$

The boundary condition on the external boundary (motor case and inserts) is, from Eq. (6),

$$\phi = 0 \tag{10}$$

The boundary condition on the internal surface (stress-free surface) is, from Eqs. (6, 8, and 9),

$$\partial\phi/\partial n = 0 \tag{11}$$

where n is the outward normal.

Since all of the equations of elasticity have been satisfied, the assumptions in Eq. (1) are justified, and this solution represents an exact solution of the problem. The stresses [Eqs. (8) and (9)] are found to be independent of the elastic shear modulus; so, from the correspondence principle,[2] Eqs. (6–11) apply for any linear viscoelastic material. Also, because of the linearity of the problem, the stresses found from this loading can be added to the stresses due to other types of loading.

Equation (7) is the familiar Poisson's equation that occurs in many physical problems, such as torsion, heat transfer, and pressurized membrane. There are several methods of solution, among them, relaxation techniques.[3] A good physical picture can be visualized by application of the well-known membrane analogy.

Consider a flat horizontal plate having a hole of the same shape as the rigid external boundary of the propellant cross section shown in Fig. 1 and an internal vertical smooth wall of the same shape as the star. The deflection of a pressurized soap film placed over this hole is governed by the same differential equation as Eq. (7), and its boundary conditions are the same as Eqs. (10) and (11). The slope of the soap film surface in any direction is, then, proportional to the shear stress in that direction. From Fig. 1 and from the membrane

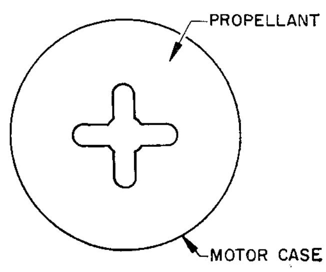

Fig. 2 Standard rocket: cross section.

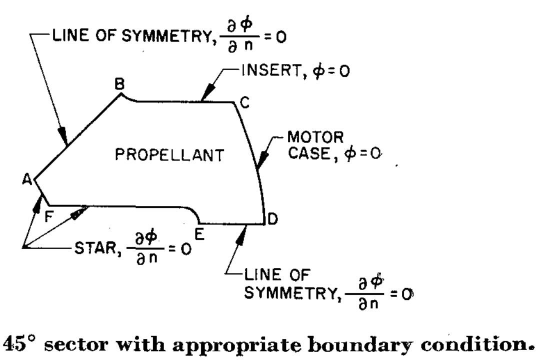

Fig. 3 A 45° sector with appropriate boundary condition.

analogy, it can be seen immediately that, if the insert has a finite angle at the re-entrant corner B, the stress will have a singularity at that point. Consequently, rounding these corners is imperative.

The symmetry of the cross section in Fig. 1 can be used to reduce the solution domain to a 45° sector, as shown in Fig. 3. The appropriate boundary conditions are also shown in this figure.

Conclusion

This problem is one of the simplest cases of antiplane elastic systems[4]; however, its solution appears to have been omitted from the literature on these systems. The reason for this apparent omission is not that it is difficult, but that it rarely occurs in applications. As was just shown, however, it is of great importance in the design of high-acceleration solid propellant rockets.

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Comments

Comments on "Dynamic Response of an Elastic Plate to a Moving Line Load"

William E. Thompson*
Cornell Aeronautical Laboratory Inc., Buffalo, N. Y.

In a recent investigation,[1] a problem very similar to that reported by Reismann[2] was studied. Although only a moderate effort was expended, it did extend over a considerable period of time so that most of the pertinent literature on the subject was ultimately identified. Without making an extensive list of references, two matters stand out. First, Mindlin[3] has stated that, except for constants, the equation of motion of the elastic plate strip (plate in plane strain) and the equation of motion of a simple beam are identical. Thus, the solutions for the dynamic response of a beam subject to a moving line load and supported in comparable ways become available for use on the corresponding plate problems. Second, in view of this, the work of Ludwig,[4] Kenney,[5] Mathews,[6] and Fryba[7] contains most of the essential results now reported anew by Reismann.

It is true that the work of Kenney and Mathews involves a linear restoring force term in the equation of motion due to the influence of a continuous elastic foundation, but a number of simplified, special cases among the several references allow one to anticipate Reismann's results.

It is regrettable that the author has not acknowledged this earlier work, especially in view of the recent, related paper by Lloyd and Miklowitz[8] in which both the references and the close association of the beam and plate problems were called to the reader's attention.

Received March 24, 1963; revision received July 12, 1963.
* Principal Mechanical Engineer. Member AIAA.

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Reply by Author to W. E. Thompson

Herbert Reismann*
Martin Company, Denver, Colo.

The paper under discussion was taken from a more complete research report,[2] which gave a more detailed treatment and also discussed the connection of Ref. 1 with the subject of beam vibration. Appended to the research report[2] is a rather complete Bibliography pertaining to the beam problem. Since the paper dealt with the subject of plates and because of the usual space restriction, only references pertaining to plates were given in Ref. 1.

Thompson implies that the solution of the problem of the beam on an elastic foundation subject to moving load will yield the results of Ref. 1. Although there are certain similarities, there are, at least, two obvious facts that indicate major differences:

1) The static wave length of an infinite beam on an elastic foundation is finite, whereas that of a plate strip is infinite (in the long direction).

2) The plate has a denumerable infinity of critical speeds, whereas the beam has only one.

Moreover, the plate of Ref. 1 is not in a state of plane strain, and therefore Mindlin's analogy, referred to by Thompson, is not applicable. It should be pointed out that the results of Ludwig (Ref. 3, cited by Thompson) are in error for the case of supercritical speeds. Finally, it would have been impossible to acknowledge the work of Lloyd and Miklowitz (Ref. 5, cited by Thompson) inasmuch as it appeared in September 1962, whereas the paper under discussion[1] was submitted in August 1962.

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